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Formulas on Queues in Burst Processes—II

By M. M. SONDHI, B. GOPINATH, and DEBASIS MITRA*

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Queues arising in buffers due to either random interruptions of the channel or variable source rates are analyzed in the framework of a single digital system. Two motivating applications are: (i) multiplexing of data with speech on telephone channels and (ii) buffering of data generated by the coding of moving images in Picturephone® *service.*

In the model a source feeds data to a buffer at a uniform rate. The buffer's access to a channel with fixed maximum rate of transmission is controlled by a switch; only when the switch is closed ("on") is the buffer able to discharge. The on-off sequence of the switch is indicated by a burst process which is a key element in this paper. In such a process, long periods during which the switch stays closed alternate with periods, called bursts, during which the on-off sequence is a first-order Markov process. The length of a burst is randomly distributed. This is a generalization of the memoryless burst process considered in an earlier paper.[1] In that paper we gave formulas for the efficient computation of various functionals of the queues arising in the system. Now we extend these formulas to hold for the generalized class of burst processes.

I. INTRODUCTION

In a recent paper[1] we considered the problem of buffering the output of a uniform source whose access to a given transmission channel is controlled by a burst process. We gave formulas for efficiently computing various functionals of queues that form in such a communication system when the controlling burst process is memoryless.

In the present paper we generalize the controlling process to one which is first-order Markov within a burst. This generalization considerably increases the usefulness of the formulas. Consider, for example, the two motivating applications discussed in Ref. 1: (*i*) multi-

* The sequence of names was decided by coin tossing.

plexing of data with speech on telephone channels[2–6] and (*ii*) buffering of data generated by the coding of moving images in *Picturephone®* service.[7] For the first application, analysis of data shows[2] that it is necessary to go to a first-order Markov process to adequately model the burst phenomena in speech signals. In the *Picturephone* application, although the correlation of data rates within a frame is negligible, it is quite significant from frame to frame.[8] For frame-to-frame coding, therefore, the present model with memory becomes necessary.

The system under consideration is shown in Fig. 1. The source emits data uniformly at the rate of 1 symbol per unit time. The transmission rate of the channel is $(k + 1)$ symbols per unit time, where k is some positive integer. The on-off pattern of the switch is indicated by a binary burst process: $E(j)$ is either 0 or 1 for $j = 0, 1, 2, \cdots$. If $E(j) = 0$ the switch is closed for the time duration $[j, j + 1)$; otherwise, the switch is open. We assume that there are long periods during which $E(j) = 0$ and that at the end of every such period the buffer is empty. The activity separated by such periods we call a burst. We assume bursts to be independent of each other, and the burst length to have a probability distribution which is either geometric or is a weighted sum of geometric distributions. Within a burst, $\{E(j)\}$ is assumed to be a homogeneous two-state Markov chain with transition probabilities θ_1 and θ_2 given by

$$\theta_1 \triangleq \text{Prob. } \{E(j + 1) = 1 | E(j) = 0\} \tag{1a}$$

$$\theta_2 \triangleq \text{Prob. } \{E(j + 1) = 0 | E(j) = 1\}, \qquad j = 0, 1, 2, \cdots. \tag{1b}$$

These two parameters completely specify the Markov chain; the probabilities of the other two possible transitions are, of course, given by

$$1 - \theta_1 = \text{Prob. } \{E(j + 1) = 0 | E(j) = 0\}$$

and

$$1 - \theta_2 = \text{Prob. } \{E(j + 1) = 1 | E(j) = 1\}.$$

We shall assume that $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$. If $\theta_1 + \theta_2 = 1$, E_j becomes a Bernoulli sequence of independent random variables, which is the case treated in Ref. 1.

In subsequent sections of this paper we will obtain the results summarized below.

In (*i*), (*ii*), and (*iii*), we assume the switch to be controlled by an infinitely long sequence generated by the Markov chain described by (1); these three results are therefore of interest in situations where the distribution of burst lengths is not known accurately.

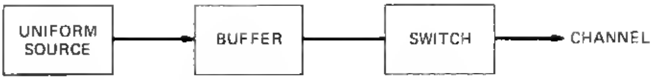

Fig. 1—Switched communication system.

(*i*) We derive a recursive formula for the steady-state distribution of buffer content for finite buffers, the recursion being with respect to the buffer size, N.

(*ii*) Let $T^{(N)}$ be the steady-state probability of a buffer of size N being full when the channel is inaccessible. ($T^{(N)}$, therefore, is the steady-state probability of a transmission fault.) We show that

$$\frac{1}{T^{(N+k+1)}} = \frac{1}{1-\theta_2}\frac{1}{T^{(N+k)}} + \frac{1-\theta_1-\theta_2}{1-\theta_2}\frac{1}{T^{(N+1)}} - \frac{1-\theta_1}{1-\theta_2}\frac{1}{T^{(N)}},$$

where $(k+1)$, as previously defined, is the transmission rate of the channel. We show that the steady-state probability of the buffer being full is $T^{(N)}/(1-\theta_2)$, and therefore satisfies the same recursive relation.

(*iii*) For a buffer of size greater than N, let $F^{(N)}$ denote the mean time to first passage through the level N. We show that $F^{(N)}$ satisfies the recursion

$$F^{(N+k+1)} = \frac{1}{1-\theta_2}F^{(N+k)} + \frac{1-\theta_1-\theta_2}{1-\theta_2}F^{(N+1)} - \frac{1-\theta_1}{1-\theta_2}F^{(N)} + \frac{\theta_1+\theta_2}{1-\theta_2}.$$

The next two results are of interest when the distribution of burst lengths is well-approximated by a weighted sum of geometric distributions.

(*iv*) Let $G^{(N)}$ be the probability of overflow for a buffer of size N *during a burst*. Then if the burst lengths have a geometrical probability distribution with parameter ρ {i.e., Prob. (burst length $= i$) $= \rho^{i-1}(1-\rho)$}, we show that

$$\frac{1}{G^{(N+k+1)}} = \frac{1}{\rho(1-\theta_2)}\frac{1}{G^{(N+k)}} + \frac{\rho(1-\theta_1-\theta_2)}{1-\theta_2}\frac{1}{G^{(N+1)}} - \frac{1-\theta_1}{1-\theta_2}\cdot\frac{1}{G^{(N)}}.$$

This result generalizes to the case when the burst length distribution is a sum of geometric distributions.

(*v*) We derive a closed expression as well as a recursive formula for the mean time for first passage through a level N *during a burst*

conditioned on the occurrence of an overflow. The recursion is with respect to N, and the bursts are assumed to be distributed as in (*iv*).

(*vi*) We determine the asymptotic behavior of all the formulas in (*i*) to (*v*) as $N \to \infty$. For instance, we prove that, as $N \to \infty$, $(1/G^{(N)}) \sim s^N$, where s is the unique positive real root of a particular polynomial, such that $s > 1/\rho > 1$.

The closed expressions are all valid for $k \geqq 1$ and $N \geqq 0$, and the recursions as stated above are valid for $N \geqq 0$. The recursive formulas provide very efficient means for computation of the various functionals, particularly in design studies where a whole range of buffer sizes is to be investigated.

1.1 Notation

Whenever necessary we will use a superscript in parentheses, e.g., $x^{(M)}$, to indicate that the quantity corresponds to a buffer of size M (or to the level M in a buffer of size greater than M). If $\mathbf{x}$ is a vector, then the superscript (M) will also indicate that the vector $\mathbf{x}^{(M)}$ is $(M + 1)$-dimensional with components $x_i^{(M)}$, $i = 0, 1, 2, \cdots, M$. These two uses of the superscript are consistent because the dimensions of all vectors defined in this paper are related to buffer size (level) in this manner. Whenever the superscript is missing, the standard value (N) will be implied.

We will use lower-case boldface letters to denote column vectors, upper-case boldface letters to denote matrixes, and a superscript T to denote the transpose. We will denote by $\mathbf{I}$ the identity matrix, by $\mathbf{1}$ the vector whose components are all equal to 1, and by $\mathbf{e}_j$ the vector whose jth component is 1 and the rest 0, e.g., $\mathbf{e}_0^T = (1, 0, \cdots, 0)$.

II. EQUATIONS OF THE PROCESSES

Let $B(t)$ be the number of symbols in the buffer at time t. Then for a buffer of size N

$$B(t + 1) = \mathrm{Max}\,[B(t) - k, 0] \quad \text{if} \quad E(t) = 0 \tag{2a}$$

$$= \mathrm{Min}\,[B(t) + 1, N] \quad \text{if} \quad E(t) = 1. \tag{2b}$$

In the last equation the assumption is that if the channel is inaccessible and the buffer is full, then the current source symbol is discarded and the buffer remains full.

In order to study the evolution of the buffer content process, it is convenient to introduce two $(N + 1)$-dimensional vectors $\mathbf{p}(t)$

$= \{p_o(t), \cdots, p_N(t)\}$ and $\mathbf{q}(t) = \{q_o(t), \cdots, q_N(t)\}$ defined by the equations

$$p_i(t) \triangleq \Pr\{B(t) = i, E(t) = 0\}, \qquad i = 0, \cdots, N \tag{3a}$$

$$q_i(t) \triangleq \Pr\{B(t) = i, E(t) = 1\}, \qquad i = 0, \cdots, N. \tag{3b}$$

Under the assumption that $\{E(t)\}$ is the two-state Markov chain defined by (1), it is straightforward to show that $\mathbf{p}(t)$ and $\mathbf{q}(t)$ represent a $2(N + 1)$-state homogeneous Markov chain. For

$$\begin{aligned} p_0(t+1) &\triangleq \Pr\{B(t+1) = 0, E(t+1) = 0\} \\ &= \sum_{i=0}^{k} \Pr\{B(t) = i, E(t+1) = 0, E(t) = 0\} \\ &= \sum_{i=0}^{k} \Pr\{E(t+1) = 0 | E(t) = 0, B(t) = i\} \\ &\qquad \times \Pr\{B(t) = i, E(t) = 0\} \\ &= (1 - \theta_1) \sum_{i=0}^{k} p_i(t), \end{aligned} \tag{4}$$

where the last step follows from the Markov property of $\{E(t)\}$. Similarly,

$$\begin{aligned} p_i(t+1) &= (1 - \theta_1)p_{i+k}(t) + \theta_2 q_{i-1}(t), \qquad i = 1, 2, \cdots, N - k, \\ &= \theta_2 q_{i-1}(t), \qquad i = N - k + 1, \cdots, N - 1, \\ &= \theta_2\{q_{i-1}(t) + q_N(t)\}, \qquad i = N. \end{aligned} \tag{5}$$

Also

$$\begin{aligned} q_i(t+1) &= \theta_1 \sum_{j=0}^{k} p_j(t), \qquad i = 0, \\ &= \theta_1 p_{i+k}(t) + (1 - \theta_2)q_{i-1}(t), \qquad i = 1, 2, \cdots, N - k, \\ &= (1 - \theta_2)q_{i-1}(t), \qquad i = N - k + 1, \cdots, N - 1, \\ &= (1 - \theta_2)\{q_{i-1}(t) + q_i(t)\}, \qquad i = N. \end{aligned} \tag{6}$$

Equations (4), (5), and (6) can be written conveniently in matrix notation as

$$\mathbf{p}(t+1) = (1 - \theta_1)\mathbf{B}\mathbf{p}(t) + \theta_2\tilde{\mathbf{A}}\mathbf{q}(t) \tag{7a}$$

$$\mathbf{q}(t+1) = \theta_1\mathbf{B}\mathbf{p}(t) + (1 - \theta_2)\tilde{\mathbf{A}}\mathbf{q}(t). \tag{7b}$$

Here the $(N+1) \times (N+1)$ matrixes $\mathbf{B}$ and $\tilde{\mathbf{A}}$ are defined as

$$\mathbf{B} \triangleq \begin{array}{cl} \overbrace{\hphantom{1 \quad 1 \cdots 1}}^{(k+1)} & \\ \begin{bmatrix} 1 & 1 \cdots 1 & & & 0 \\ & & 1 & & \\ & & & \ddots & \\ & 0 & & & 1 \\ & & & & \\ & & & & \end{bmatrix} & \begin{matrix} 0 \\ 1 \\ \vdots \\ N-k, \\ \vdots \\ N \end{matrix} \end{array} \qquad \tilde{\mathbf{A}} \triangleq \begin{bmatrix} 0 & & & & 0 \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & 0 & & 1 & 1 \end{bmatrix}. \tag{8}$$

Notice that the composite matrix

$$\begin{bmatrix} (1-\theta_1)\mathbf{B} & \theta_2\tilde{\mathbf{A}} \\ \theta_1\mathbf{B} & (1-\theta_2)\tilde{\mathbf{A}} \end{bmatrix} \tag{9}$$

is stochastic (nonnegative elements and every column sums to 1) and independent of t. Equations (7a), (7b) are, therefore, the transition equations of a $2(N+1)$-state homogeneous Markov chain.

2.1 Equations for some new probabilities

For many of the derivations in the succeeding sections (e.g., mean first passage time, probability of no overflow, etc.) it is convenient to define certain new probabilities $r_i(t)$ and $s_i(t)$, $i = 0, 1, \cdots, N$. Consider a buffer of size greater than N and let $X(t)$ be the event $\bigcap_{s=0}^{t} \{B(s) \leq N\}$, i.e., the event that $B(s)$ does not exceed N at any of the time instants $s = 0, 1, 2, \cdots, t$. Then

$$r_i(t) \triangleq \Pr\{B(t) = i, E(t) = 0, X(t)\}, \qquad i = 0, \cdots, N, \tag{10a}$$

$$s_i(t) \triangleq \Pr\{B(t) = i, E(t) = 1, X(t)\}, \qquad i = 0, \cdots, N. \tag{10b}$$

We define the $(N+1)$-dimensional vectors $\mathbf{r}(t)$ and $\mathbf{s}(t)$ with components $\{r_o(t), \cdots, r_N(t)\}$ and $\{s_o(t), \cdots, s_N(t)\}$, respectively.

In a manner analogous to the derivation of eqs. (7a) and (7b), we can derive recurrence relations giving $\mathbf{r}(t+1)$, $\mathbf{s}(t+1)$ in terms of $\mathbf{r}(t)$, $\mathbf{s}(t)$. Thus, for $i = 0, 1, \cdots, N$,

$$\begin{aligned} r_i(t+1) &\triangleq \Pr\{B(t+1) = i, E(t+1) = 0, X(t+1)\} \\ &= \Pr\{B(t+1) = i, E(t+1) = 0, X(t)\} \\ &= (1-\theta_1)\Pr\{B(t+1) = i, E(t) = 0, X(t)\} \\ &\qquad + \theta_2 \Pr\{B(t+1) = i, E(t) = 1, X(t)\}, \end{aligned} \tag{11}$$

where the last equation follows from the Markov property of $\{E(t)\}$. As before, $B(t+1)$ and $E(t)$ determine the possible values of $B(t)$

and we get

$$
\begin{aligned}
r_i(t+1) &= (1-\theta_1)\sum_{j=0}^{k} r_j(t), \qquad i = 0,\\
&= (1-\theta_1)r_{i+k}(t) + \theta_2 s_{i-1}(t), \qquad i = 1, 2, \cdots, N-k,\\
&= \theta_2 s_{i-1}(t), \qquad i = N-k+1, \cdots, N.
\end{aligned} \tag{12}
$$

Comparison of eq. (12) with eqs. (4) and (5) shows that for $t = 0, 1, \cdots,$

$$\mathbf{r}(t+1) = (1-\theta_1)\mathbf{Br}(t) + \theta_2\mathbf{As}(t), \tag{13}$$

where $\mathbf{A}$ is obtained from $\tilde{\mathbf{A}}$ by setting to 0 the single nonzero entry on its main diagonal, i.e.,

$$\mathbf{A} = \tilde{\mathbf{A}} - \mathbf{e}_N\mathbf{e}_N^T. \tag{14}$$

Analogously to (13) we can also show that

$$\mathbf{s}(t+1) = \theta_1\mathbf{Br}(t) + (1-\theta_2)\mathbf{As}(t). \tag{15}$$

The transition equations (13) and (15), although very similar to eqs. (7a) and (7b), differ fundamentally from them in that $\mathbf{A}$, and consequently the matrix

$$\begin{bmatrix} (1-\theta_1)\mathbf{B} & \theta_2\mathbf{A} \\ \theta_1\mathbf{B} & (1-\theta_2)\mathbf{A} \end{bmatrix}, \tag{16}$$

are not stochastic.

We close this section by deriving from (13) and (15) a useful second-order recursion involving $\mathbf{s}(t+2)$, $\mathbf{s}(t+1)$, and $\mathbf{s}(t)$. Multiplying (13) by θ_1, (15) by $(\theta_1 - 1)$, and adding we get

$$\theta_1\mathbf{r}(t+1) = (1-\theta_1)\mathbf{s}(t+1) - (1-\theta_1-\theta_2)\mathbf{As}(t). \tag{17}$$

From (15),

$$\mathbf{s}(t+2) = \theta_1\mathbf{Br}(t+1) + (1-\theta_2)\mathbf{As}(t+1). \tag{18}$$

Premultiplying (17) by $\mathbf{B}$ and adding to (18) gives

$$\mathbf{s}(t+2) = [(1-\theta_1)\mathbf{B} + (1-\theta_2)\mathbf{A}]\mathbf{s}(t+1) - (1-\theta_1-\theta_2)\mathbf{BAs}(t), \qquad t = 0, 1, 2, \cdots. \tag{19}$$

As we will have to refer frequently to the recursion (19) it is convenient to define

$$\mathbf{C} \triangleq [(1-\theta_1)\mathbf{B} + (1-\theta_2)\mathbf{A}]$$

and

$$\mathbf{D} \triangleq -(1-\theta_1-\theta_2)\mathbf{BA} \tag{20}$$

so that eq. (19) becomes

$$\mathbf{s}(t+2) = \mathbf{C}\mathbf{s}(t+1) + \mathbf{D}\mathbf{s}(t), \qquad t = 0, 1, 2, \cdots. \tag{21}$$

III. INFINITELY LONG SEQUENCES

When the burst length distribution is not known, useful information can still be obtained by considering the behavior of the buffer content when the switch in Fig. 1 is controlled by infinitely long sequences generated by the Markov chain (1). In this section we derive various functionals for such a situation.

3.1 Stationary distributions for finite buffers

In eqs. (7a), (7b), if we set $\mathbf{p}(t+1) = \mathbf{p}(t) = \mathbf{p}$ and $\mathbf{q}(t+1) = \mathbf{q}(t) = \mathbf{q}$, then the vectors $\mathbf{p} = \{p_0, \cdots, p_N\}$ and $\mathbf{q} = \{p_0, \cdots, q_N\}$ give the limiting distributions[9] as $t \to \infty$ of the buffer content process defined in Section II. The limiting distributions $\mathbf{p}$, $\mathbf{q}$ are thus the solutions of

$$\mathbf{p} = (1-\theta_1)\mathbf{B}\mathbf{p} + \theta_2\tilde{\mathbf{A}}\mathbf{q} \tag{22a}$$

$$\mathbf{q} = \theta_1\mathbf{B}\mathbf{p} + (1-\theta_2)\tilde{\mathbf{A}}\mathbf{q} \tag{22b}$$

with, of course, the normalization

$$\mathbf{1}^T(\mathbf{p}+\mathbf{q}) = 1. \tag{23}$$

In this section we derive a simple formula for computing the vectors $\mathbf{p}$ and $\mathbf{q}$ for a given buffer size $(N+1)$ in terms of p and q for a buffer of size N. As a first step we simplify the problem by eliminating p from eqs. (22a), (22b). Multiplying (22a) by θ_1 and (22b) by $(\theta_1 - 1)$ and adding gives

$$\mathbf{p} = \left(\frac{1-\theta_1}{\theta_1}\right)\mathbf{q} - \frac{1-\theta_1-\theta_2}{\theta_1}\tilde{\mathbf{A}}\mathbf{q}. \tag{24}$$

Substituting (24) into (22b) gives

$$[\mathbf{I} - (1-\theta_1)\mathbf{B} - (1-\theta_2)\tilde{\mathbf{A}} + (1-\theta_1-\theta_2)\mathbf{B}\tilde{\mathbf{A}}]\mathbf{q} = 0. \tag{25}$$

Premultiplying (24) by $\mathbf{1}^T$ and subtracting from (23) gives

$$\mathbf{1}^T\mathbf{q} = \frac{\theta_1}{\theta_1+\theta_2} \tag{26}$$

since $\mathbf{1}^T\tilde{\mathbf{A}} = \mathbf{1}^T$. It is important to note that the $N+1$ component equations in (25) are not independent. Indeed, since $\mathbf{1}^T\tilde{\mathbf{A}} = \mathbf{1}^T\mathbf{B} = \mathbf{1}^T$,

it is clear that the first equation is just the sum of the rest and may therefore be ignored. The remaining N equations are linearly independent and we can solve them for $q_0, \cdots, q_{N-1}$ in terms of q_N, and then obtain q_N from (26). Finally, we can obtain p from (24).

In carrying out the solution of (25) and (26) in this manner the recursion we are looking for becomes obvious if we define the $(N + 1)$-dimensional vector $\mathbf{y}^{(N)}$ with components given by*

$$y_i^{(N)} = q_{N-i}^{(N)}/q_N^{(N)}, \qquad i = 0, \cdots, N. \tag{27}$$

[The meaning of the superscript (N) is given in Section 1.1.] Equations (25) and (27) give

$$y_0^{(N)} = 1 \tag{28a}$$

$$y_1^{(N)} = \frac{\theta_2}{1 - \theta_2} \tag{28b}$$

$$y_i^{(N)} = \frac{y_{i-1}^{(N)}}{1 - \theta_2}, \qquad i = 2, \cdots, k \tag{28c}$$

$$y_{k+1}^{(N)} = \frac{y_k^{(N)}}{1 - \theta_2} + \frac{1 - \theta_1 - \theta_2}{1 - \theta_2} y_1^{(N)} - \frac{\theta_2}{1 - \theta_2} \tag{28d}$$

$$y_i^{(N)} = \frac{y_{i-1}^{(N)}}{1 - \theta_2} + \frac{1 - \theta_1 - \theta_2}{1 - \theta_2} y_{i-k}^{(N)} - \frac{1 - \theta_1}{1 - \theta_2} y_{i-k-1}^{(N)}, \quad i > k + 1. \tag{28e}$$

The important fact about (28) is that the superscript (N) is superfluous. If N is changed to $N + 1$, for instance, in (28) we see that

$$y_i^{(N+1)} = y_i^{(N)}, \qquad i = 0, \cdots, N, \tag{29}$$

and the last component of $y^{(N+1)}$ is

$$y_{N+1}^{(N+1)} = \frac{1}{1 - \theta_2} y_N^{(N)} + \frac{1 - \theta_1 - \theta_2}{1 - \theta_2} y_{N-k+1}^{(N)} - \frac{1 - \theta_1}{1 - \theta_2} y_{N-k}^{(N)}. \tag{30}$$

Thus the vector $\mathbf{y}^{(N+1)}$ is obtained from $\mathbf{y}^{(N)}$ by merely appending to the components of $\mathbf{y}^{(N)}$ one component given by (30). To complete the recursion for $\mathbf{q}^{(N+1)}$, we note from (26) and (27) that

$$\frac{1}{q_{N+1}^{(N+1)}} = \left(\frac{\theta_1 + \theta_2}{\theta_1}\right) \sum_{i=0}^{N+1} y_i^{(N+1)} \tag{31}$$

* Note that $q_N^{(N)} \neq 0$, for otherwise the solution q of (25) is the null vector which cannot satisfy (26).

and therefore, from (29) and (30),

$$\begin{aligned}\frac{1}{q_{N+1}^{(N+1)}} &= \frac{1}{q_N^{(N)}} + \frac{\theta_1+\theta_2}{\theta_1}\, y_{N+1}^{(N+1)} \\ &= \frac{1}{q_N^{(N)}} + \frac{\theta_1+\theta_2}{\theta_1(1-\theta_2)} \\ &\qquad \times \left[y_N^{(N)} + (1-\theta_1-\theta_2)y_{N+1-k}^{(N)} - (1-\theta_1)y_{N-k}^{(N)}\right]. \end{aligned} \tag{32}$$

Equation (32) gives $q_{N+1}^{(N+1)}$ in terms of the components of $\mathbf{q}^{(N)}$.

3.2 Probability of transmission fault and of buffer being full

Frequently it is adequate to determine the variation with buffer size of the components $p_N^{(N)}$ and $q_N^{(N)}$ rather than of the complete distributions $\mathbf{p}^{(N)}$ and $\mathbf{q}^{(N)}$. Notice that the probability of transmission fault $T^{(N)}$ is, by the definition given in Section I, identical to $q_N^{(N)}$; and the probability that a buffer of size N is full is clearly $p_N^{(N)} + q_N^{(N)}$. It is therefore of interest to obtain recursions for these quantities without having to compute the entire p and q vectors from the recursions derived in Section 3.1.

By premultiplying eqs. (22a) and (22b) by $\mathbf{e}_N^T(\triangleq\{0, 0, \cdots, 0, 1\})$ we get

$$\mathbf{e}_N^T\mathbf{p} = \theta_2\mathbf{e}_N^T\tilde{\mathbf{A}}\mathbf{q} = \frac{\theta_2}{1-\theta_2}\,\mathbf{e}_N^T\mathbf{q} \tag{33}$$

or

$$\mathbf{e}_N^T(\mathbf{p}+\mathbf{q}) = \frac{\mathbf{e}_N^T\mathbf{q}}{1-\theta_2} = \frac{T^{(N)}}{1-\theta_2}, \tag{34}$$

i.e.,

$$p_N^{(N)} + q_N^{(N)} = \frac{1}{1-\theta_2}\cdot T^{(N)}.$$

It therefore suffices to obtain a recursion for $T^{(N)}$. Suppressing the superscript (N) from (28e), and summing over the index i from $k+2$ to $N+k+1$, we get

$$\sum_{i=k+2}^{N+k+1} y_i = \frac{1}{1-\theta_2}\sum_{i=k+1}^{N+k} y_i + \frac{1-\theta_1-\theta_2}{1-\theta_2}\sum_{i=2}^{N+1} y_i - \frac{1-\theta_1}{1-\theta_2}\sum_{i=1}^{N} y_j. \tag{35}$$

Since $T^{(N)} = q_N^{(N)}$, (31) is used to relate $T^{(N)}$ to $\{y_i\}$. Now substituting

the values of $\{y_i\}$ given in (28) we ohtain

$$\frac{1}{T^{(N+k+1)}} - \frac{1}{1-\theta_2}\frac{1}{T^{(N+k)}} - \frac{1-\theta_1-\theta_2}{1-\theta_2}\frac{1}{T^{(N+1)}} + \frac{1-\theta_1}{1-\theta_2}\frac{1}{T^{(N)}} = 0, \qquad N \geqq 1. \tag{36}$$

Equation (36) is the reeursion quoted in Seetion I.

3.3 Mean first passage time

Let N be a positive integer and let the huffer be of size greater than N. Let an infinitely long burst start at $t = 0$, with the buffer initially empty, and let $F^{(N)}$ denote the mean time required for the buffer content to first exeeed N. The manner in which $F^{(N)}$ depends on N is a useful guide in designing an adequate buffer, espeeially when the distrihution of burst lengths is not aeeurately known. In this section we derive a recursive formula for $F^{(N)}$, the reeursion heing with respeet to the level N.

By definition, the Nth eomponent of the veetor $\mathbf{s}(t)$ defined in eq. (10h) is the probahility that the level N is exceeded for the first time at the instant $t + 1$. Therefore,

$$\begin{aligned} F^{(N)} &= \sum_{t=0}^{\infty} (t+1) s_N(t) \\ &= \mathbf{e}_N^T \sum_{t=0}^{\infty} (t+1)\mathbf{s}(t). \end{aligned} \tag{37}$$

In the appendix we show that if λ is an eigenvalue of the matrix defined in (16), then $|\lambda| < 1$. This proves the eonvergenee of the series in (37).

We proceed by obtaining an expression for $\sum_{t=0}^{\infty} (t+1)\mathbf{s}(t)$ by the method of generating functions. Let

$$\mathbf{S}(z) \triangleq \sum_{t=0}^{\infty} z^{t+1}\mathbf{s}(t) \tag{38}$$

so that

$$\mathbf{S}'(z) = \sum_{t=0}^{\infty} (t+1) z^t \mathbf{s}(t) \tag{39}$$

and, in partieular,

$$\mathbf{S}'(1) = \sum_{t=0}^{\infty} (t+1)\mathbf{s}(t). \tag{40}$$

From the equation, (21), governing the evolution of $\{s(t)\}$ we find that

$$\mathbf{S}(z) = [I - z\mathbf{C} - z^2\mathbf{D}]^{-1}\{z\mathbf{s}(0) + z^2\mathbf{s}(1) - z^2\mathbf{C}\mathbf{s}(0)\}. \quad (41)$$

It is shown in the appendix that the above matrix inverse exists for all $|z| \leqq 1$. Following the procedure already outlined [eqs. (39) and (40)] we find that

$$\sum_{t=0}^{\infty}(t+1)\mathbf{s}(t) = [\mathbf{I}-\mathbf{C}-\mathbf{D}]^{-1}[\mathbf{C}+2\mathbf{D}][\mathbf{I}-\mathbf{C}-\mathbf{D}]^{-1}\{\mathbf{s}(0)+\mathbf{s}(1)-\mathbf{C}\mathbf{s}(0)\}$$
$$+[\mathbf{I}-\mathbf{C}-\mathbf{D}]^{-1}\{\mathbf{s}(0)+2\mathbf{s}(1)-2\mathbf{C}\mathbf{s}(0)\}. \quad (42)$$

The resulting expression for $F^{(N)}$, from (37) and (42), is further simplified by using the following identities:

$$\mathbf{e}_N^T = \frac{1}{\theta_1}\mathbf{1}^T[\mathbf{I} - \mathbf{C} - \mathbf{D}],$$

and

$$\mathbf{1}^T[\mathbf{C} + 2\mathbf{D}] = (\theta_1 + \theta_2)\mathbf{1}^T + (1 - \theta_2 - 2\theta_1)\mathbf{e}_N^T.$$

Then

$$F^{(N)} = \frac{\theta_1 + \theta_2}{\theta_1}\cdot\mathbf{1}^T[\mathbf{I} - \mathbf{C} - \mathbf{D}]^{-1}\{\mathbf{s}(0) - (1 - \theta_1)\mathbf{B}\mathbf{s}(0) + \theta_1\mathbf{B}\mathbf{r}(0)\}$$
$$+ (\mathbf{1}^T\mathbf{r}(0) - \theta_2)/\theta_1. \quad (43)$$

The above expression for $F^{(N)}$ holds for arbitrary initial states of the buffer. However, as mentioned in the beginning of this section, in deriving a recursive formula for $F^{(N)}$ we will assume the buffer empty at $t = 0$. In that case, $\mathbf{r}(0) = \tau\mathbf{e}_0$ and $\mathbf{s}(0) = (1 - \tau)\mathbf{e}_0$ with $\tau\epsilon[0, 1]$. Substituting in (43) we get, for this special case,

$$F^{(N)} = (\theta_1 + \theta_2)\mathbf{1}^T(\mathbf{I} - \mathbf{C} - \mathbf{D})^{-1}\mathbf{e}_0 + \frac{\tau - \theta_2}{\theta_1}. \quad (44)$$

We can derive a recursion for the quantity

$$f^{(N)} \triangleq \mathbf{1}^T(\mathbf{I} - \mathbf{C} - \mathbf{D})^{-1}\mathbf{e}_0 \quad (45)$$

from which the recursion for $F^{(N)}$ will follow immediately. The procedure is very similar to the one used to derive (36). Thus let $\mathbf{x}^T = (x_0, x_1, \cdots, x_N)$ be the solution of

$$(\mathbf{I} - \mathbf{C} - \mathbf{D})\mathbf{x} = \mathbf{e}_0. \quad (46)$$

Then, since $\mathbf{1}^T(\mathbf{I} - \mathbf{C} - \mathbf{D}) = \theta_1\mathbf{e}_N^T$, we get $x_N = 1/\theta_1$. We may re-

place the first of the component equations in (46) hy this relation. Exactly as in (27) and (28), we find that the components $x_i^{(N)}$ $(i = 0, \cdots, N)$ of the vector $\mathbf{x}^{(N)}$ are, in reverse order, the first $N + 1$ numhers $\hat{x}_i$ in the sequence generated as follows:

$$\hat{x}_0 = \frac{1}{\theta_1} \tag{47a}$$

$$\hat{x}_i = \frac{1}{1-\theta_2}\,\hat{x}_{i-1}, \qquad i = 1, \cdots, k, \tag{47h}$$

$$\hat{x}_i = \frac{1}{1-\theta_2}\,\hat{x}_{i-1} + \frac{1-\theta_1-\theta_2}{1-\theta_2}\,\hat{x}_{i-k} - \frac{1-\theta_1}{1-\theta_2}\,\hat{x}_{i-k-1}, \qquad i > k. \tag{47c}$$

Summing (47c) over i from $k + 1$ to $N + k + 1$ and noting that $f^{(N)} = \sum_{i=0}^{N} x_i$, we get

$$\begin{aligned} f^{(N+k+1)} - \frac{1}{1-\theta_2} f^{(N+k)} - \frac{1-\theta_1-\theta_2}{1-\theta_2} f^{(N+1)} - \frac{1-\theta_1}{1-\theta_2} f^{(N)} \\ = \sum_{i=0}^{k} \hat{x}_k - \frac{1}{1-\theta_2}\sum_{i=0}^{k-1}\hat{x}_i - \frac{1-\theta_1-\theta_2}{1-\theta_2}\,\hat{x}_0 \\ = \frac{1}{1-\theta_2}, \end{aligned} \tag{48}$$

where the last step follows from (47a), (47h). However,

$$f^{(N)} = \{F^{(N)} - (\tau - \theta_2)/\theta_1\}/(\theta_1 + \theta_2).$$

Suhstituting in (48) we get

$$\begin{aligned} F^{(N+k+1)} = \frac{1}{1-\theta_2} F^{(N+k)} + \frac{1-\theta_1-\theta_2}{1-\theta_2} F^{(N+1)} - \frac{1-\theta_1}{1-\theta_2} \\ + \frac{\theta_1+\theta_2}{1-\theta_2}, \qquad N = 0, 1, 2, \cdots. \end{aligned} \tag{49}$$

Interestingly, τ does not appear explicitly in the recursion (49); it does, of course, affect the initial conditions [i. e., the values of $F^{(0)}, \cdots, F^{(k)}$] via eq. (44).

It is interesting to note that the forcing term $(\theta_1 + \theta_2)/(1 - \theta_2)$ in (49) can be eliminated. By direct substitution it is seen that if $\theta_1 \neq k\theta_2$ then $F^{(N)} - (\theta_1 + \theta_2)N/(\theta_1 - k\theta_2)$ satisfies the homogeneous recursion (49). When $\theta_1 = k\theta_2$, the same is true of $F^{(N)} - (\theta_1+\theta_2)N^2/k(2-\theta_1-\theta_2)$. These transformations which reduce (49) to the homogeneous form will be of use when we investigate the asymptotics of solutions in Section V.

IV. BURSTS WITH GEOMETRICALLY DISTRIBUTED LENGTHS

When information is available concerning the distribution of burst lengths we can compute design parameters which are more realistic than the quantities $T^{(N)}$ and $F^{(N)}$ discussed in the preceding sections. Clearly an event is of consequence only if it occurs within a hurst. Its prohability of occurrence at the tth instant must therefore he weighted by the probability that the burst length exceeds t. If tbe distrihution of burst lengths is the weighted sum of geometric distributions, i.e.,

$$\text{Prob. \{Burst length} = i\} = \sum_{k=0}^{J} \beta_k (1 - \rho_k) \rho_k^{i-1},$$
$$i = 1, 2, \cdots; \quad 0 < \rho_k < 1, \tag{50}$$

then simple recursions can be obtained for such weighted averages. To keep the derivations simple we have only treated the case $J = 1$ since, as shown in Ref. 1, generalization to higher values of J is straigbtforward. In Sections 4.1 and 4.2 we derive such recursions for tbe probability of overflow within a burst and for the mean time to first cross a level within a burst.

4.1 Overflow within a burst

For a buffer of size greater than N let $G^{(N)}$ denote the probability that the buffer content exceeds N (at least once) during a burst. It is clear that $G^{(N)}$ also equals the probability that a transmission fault occurs (at least once) during a burst, when the buffer size is N. We call $G^{(N)}$ the probability of overflow.

By its definition in (10), $s_N(t)$ is the probability that the buffer content exceeds N for the first time at $t + 1$. Therefore,

$$\begin{aligned} G^{(N)} &\triangleq \sum_{t=0}^{\infty} s_N(t) \text{ Prob. \{burst lengtb} \geqq (t+1)\} \\ &= \sum_{t=0}^{\infty} s_N(t) \rho^t \\ &= \mathbf{e}_N^T \sum_{t=0}^{\infty} \rho^t \mathbf{s}(t). \end{aligned} \tag{51}$$

As proved in the appendix, the matrix in (16) has all its eigenvalues strictly within the unit circle. Therefore the series in (51) converges for $\rho \leqq 1$.

Multiplying (21) by ρ^{t+2} and summing over t from 0 to ∞ we get, on re-arranging terms,

$$(\mathbf{I} - \rho\mathbf{C} - \rho^2\mathbf{D}) \sum_{t=0}^{\infty} \rho^t \mathbf{s}(t) = \mathbf{s}(0) + \rho\{\mathbf{s}(1) - \mathbf{C}\mathbf{s}(0)\}$$

$$= [\mathbf{I} - \rho(1 - \theta_1)\mathbf{B}]\mathbf{s}(0) + \rho\theta_1\mathbf{B}\mathbf{r}(0). \quad (52)$$

In the appendix we show that $(\mathbf{I} - \rho\mathbf{C} - \rho^2\mathbf{D})$ is nonsingular for all $\rho \leqq 1$. Therefore

$$G^{(N)} = \mathbf{e}_N^T(\mathbf{I} - \rho\mathbf{C} - \rho^2\mathbf{D})^{-1}[\{\mathbf{I} - \rho(1 - \theta_1)\mathbf{B}\}\mathbf{s}(0) + \rho\theta_1\mathbf{B}\mathbf{r}(0)]. \quad (53)$$

As before, specializing to the interesting case of an initially empty buffer, i.e., $\mathbf{r}(0) = \tau\mathbf{e}_0$, $\mathbf{s}(0) = (1 - \tau)\mathbf{e}_0$, with τ in $[0, 1]$, we get

$$G^{(N)} = [(1 - \tau)(1 - \rho) + \rho\theta_1]\mathbf{e}_N^T(\mathbf{I} - \rho\mathbf{C} - \rho^2\mathbf{D})^{-1}\mathbf{e}_0. \quad (54)$$

We can obtain a recursion for $G^{(N)}$ by a procedure almost identical to that used in obtaining the recursion for $T^{(N)}$. Note that if $\mathbf{z}^{(N)}$ is a vector such that

$$(\mathbf{I} - \rho\mathbf{C} - \rho^2\mathbf{D})\mathbf{z}^{(N)} = \mathbf{e}_0 \quad (55)$$

then the components of the vector $\mathbf{z}^{(N)}/z_N^{(N)}$ are, in reverse order, the first $N + 1$ numbers in the sequence $\hat{z}_i$, $i = 0, 1, 2, \cdots$, generated by the relations

$$\hat{z}_0 = 1 \quad (56a)$$

$$\hat{z}_i = \frac{1}{\rho(1 - \theta_2)} \hat{z}_{i-1} \qquad i = 1, \cdots, k, \quad (56b)$$

$$\hat{z}_i = \frac{1}{\rho(1 - \theta_2)} \hat{z}_{i-1} + \frac{\rho(1 - \theta_1 - \theta_2)}{1 - \theta_2} \hat{z}_{i-k} - \frac{1 - \theta_1}{1 - \theta_2} \hat{z}_{i-k-1}, \qquad i > k. \quad (56c)$$

The first component equation in (55) then gives

$$\frac{1}{z_N^{(N)}} = \sum_{i=0}^{k} \pi_i \hat{z}_{N-i}, \qquad N > k, \quad (57)$$

where $\pi_0, \cdots, \pi_k$ are the leading $(k + 1)$ entries in the first row of $(\mathbf{I} - \rho\mathbf{C} - \rho^2\mathbf{D})$. (The remaining components of this row are null.) For $N > 2k$, each term on the right-hand side of (57) satisfies the recursion (56c). Therefore $1/z_N^{(N)}$ satisfies the same recursion. From (54), since $G^{(N)}$ is proportional to $z_N^{(N)}$ we find that $1/G^{(N)}$ also satisfies

the same recursion, i.e., for $N > 2k$,

$$\frac{1}{G^{(N)}} = \frac{1}{\rho(1-\theta_2)} \cdot \frac{1}{G^{(N-1)}} + \frac{\rho(1-\theta_1-\theta_2)}{1-\theta_2} \frac{1}{G^{(N-k)}} - \frac{1-\theta_1}{1-\theta_2} \frac{1}{G^{(N-k-1)}}. \tag{58}$$

It can additionally be shown that the above recursion holds for $2k \geqq N > 1$, by direct substitution of the initial values of $G^{(N)}$.

4.2 Mean time for first passage within a burst

For a buffer of size greater than N, let t denote the time required for the buffer content to first exceed N within a burst. Let $H^{(N)}$ denote the expectation of t conditional to the hypothesis that the level N is indeed exceeded within the burst. (Equivalently, $H^{(N)}$ is the mean time taken by a buffer of size N to first overflow within a burst, given that an overflow does occur.) Clearly

$$\begin{aligned} H^{(N)} &= \sum_{t=0}^{\infty} (t+1) s_N(t) \cdot [\text{Prob. that burst length} \geqq t+1]/G^{(N)} \\ &= \sum_{t=0}^{\infty} (t+1) s_N(t) \rho^t / G^{(N)} \\ &= \mathbf{e}_N^T \sum_{t=0}^{\infty} (t+1) \rho^t \mathbf{s}(t) / G^{(N)}. \end{aligned} \tag{59}$$

A comparison of (51) and (59) shows that

$$H^{(N)} = \frac{d}{d\rho} (\rho G^{(N)}) \cdot \frac{1}{G^{(N)}}. \tag{60}$$

Multiplying (53) or (54) by ρ and differentiating with respect to ρ we can get closed expressions for $H^{(N)}$ for arbitrary initial state and for the buffer initially empty. The resulting expressions are rather unwieldy.

We can also use (60) to get a recursion for $H^{(N)}$. Thus let

$$V^{(N)} \triangleq \frac{1}{\rho G^{(N)}}. \tag{61}$$

Then

$$\begin{aligned} H^{(N)} &= \frac{d}{d\rho} \left(\frac{1}{V_N} \right) \cdot \frac{1}{G^{(N)}} \\ &= -\frac{\rho U^{(N)}}{V^{(N)}}, \end{aligned} \tag{62}$$

where $U^{(N)} \triangleq (d/d\rho)V^{(N)}$. Here $V^{(N)}$ satisfies the recursion (58), and $U^{(N)}$ satisfies a recursion obtained by differentiating the recursion for $V^{(N)}$. Thus

$$V^{(N)} = \frac{1}{\rho(1-\theta_2)} V^{(N-1)} + \frac{\rho(1-\theta_1-\theta_2)}{1-\theta_2} V^{(N-k)} - \frac{1-\theta_1}{1-\theta_2} V^{(N-k-1)}$$

and

$$U^{(N)} = \frac{1}{\rho(1-\theta_2)} U^{(N-1)} + \rho\frac{(1-\theta_1-\theta_2)}{1-\theta_2} U^{(N-k)} - \frac{1-\theta_1}{1-\theta_2} U^{(N-k-1)}$$
$$- \frac{1}{\rho^2(1-\theta_2)} V^{(N)} + \frac{(1-\theta_1-\theta_2)}{1-\theta_2} V^{(N-k)}. \quad (63)$$

V. ASYMPTOTIC BEHAVIOR

In this section we discuss the behavior as $N \to \infty$ of sequences generated by the recursion

$$\varphi_N - \frac{1}{\mu(1-\theta_2)} \varphi_{N-1} - \frac{\mu(1-\theta_1-\theta_2)}{1-\theta_2} \varphi_{N-k}$$
$$+ \frac{1-\theta_1}{1-\theta_2} \varphi_{N-k-1} = \xi_N, \quad (64)$$

with $N = k+1,\ k+2,\ \cdots$ and $0 < \theta_1 < 1$, $0 < \theta_2 < 1$, and $0 < \mu \leqq 1$ the parameter ranges.

Every recursion derived in this paper can be put into the canonical form (64) by simple manipulations; furthermore, all but the recursion (63), Section 4.2, correspond to the homogeneous form of (64), i.e., $\xi_N \equiv 0$. In formulas for infinitely long burst (Sections 3.1, 3.2, 3.3) the parameter $\mu = 1$; in formulas for geometrically distributed bursts (Sections 4.1, 4.2) $0 < \mu = \rho < 1$.

Due to the linear, time-independent nature of the recursions in (64), the behavior of the solutions is determined by the sequence $\{\xi_N\}$ and the roots, λ_i, of the characteristic polynomial:

$$C(\lambda, \mu) \triangleq \mu(1-\theta_2)\lambda^{k+1} - \lambda^k - \mu^2(1-\theta_1-\theta_2)\lambda + \mu(1-\theta_1). \quad (65)$$

For the special case $\mu = 1$ the relevant properties of the roots were derived in Ref. 2. Here we derive the properties for arbitrary μ in the range $0 < \mu \leqq 1$. These properties are summarized in the following

Lemma: For the range of parameters specified above, (a) $C(\lambda, \mu)$ has exactly two positive real zeros λ_1 and λ_2 which lie in the ranges $[\mu(1-\theta_1)]$

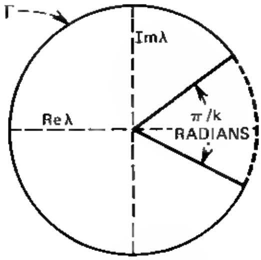

Fig. 2—Proof of lemma.

$< (\lambda_1)^k \leqq \mu$ *and* $1/\mu \leqq \lambda_2 < 1/\mu(1 - \theta_2)$ (*the equality signs are unnecessary unless* $\mu = 1$); (*b*)* *the remaining zeros all satisfy* $|\lambda_i|^k < \mu$.

Proof: (a) Regardless of the sign of $(1 - \theta_1 - \theta_2)$ there are two sign reversals in the coefficients of $C(\lambda, \mu)$. By Descartes' rule, therefore, $C(\lambda, \mu)$ has at most two positive real zeros. On the other hand, successively setting $\lambda = 0$, $\lambda^k = \mu(1 - \theta_1)$, $\lambda^k = \mu$, $\lambda = 1/\mu$, $\lambda = 1/\mu(1 - \theta_2)$ we find that $C(\lambda, \mu)$ takes on the respective values $\mu(1 - \theta_1)$, $\mu^2\theta_1\theta_2[\mu(1-\theta_1)]^{1/k}$, $-\mu\theta_1(1-\mu^{(k+1)/k})$, $-\theta_2(\mu^{-k}-\mu)$, and $\mu\theta_1\theta_2/(1-\theta_2)$. Also $C(\lambda, \mu) \rightarrow +\infty$ as $\lambda \rightarrow +\infty$. For $0 < \mu < 1$, therefore, there are exactly two zeros in the respective ranges asserted. For $\mu = 1$ further examination is required to decide whether one or both of these zeros become exactly equal to 1. Noticing that $C(1, 1) = 0$ and $(\partial/\partial\lambda) C(1, 1) = \theta_1 - k\theta_2$, it follows that when $\mu = 1$, either λ_1 or λ_2 or both become equal to 1 according as $\theta_1 - k\theta_2 < 0, > 0$, or $= 0$. (b) We will prove the stronger result that the remaining zeros lie strictly within the contour Γ (Fig. 2) defined by the following segments in the complex λ plane:

$$\lambda = Re^{j(\pi/k)} \qquad 0 \leqq R \leqq \mu^{1/k} \tag{66a}$$

$$= \mu^{1/k}e^{j\theta} \qquad \frac{\pi}{k} \leqq \theta \leqq 2\pi - \frac{\pi}{k} \tag{66b}$$

$$= Re^{-j(2\pi-\pi/k)} \qquad 0 \leqq R \leqq \mu^{1/k}. \tag{66c}$$

To prove this let us define

$$C_1 \triangleq \mu\lambda[(1 - \theta_2)\lambda^k - \mu(1 - \theta_1 - \theta_2)] \tag{67a}$$

$$C_2 \triangleq \lambda^k - \mu(1 - \theta_1) \tag{67b}$$

---

* We are tacitly assuming $k > 1$. For $k = 1$, $C(\lambda, \mu)$ becomes a quadratic with both roots positive and real in the ranges given in (a).

so that

$$C(\lambda, \mu) = C_1 - C_2$$
$$= C_2\left(\frac{C_1}{C_2} - 1\right). \tag{68}$$

We will show that $\text{Re}[C_1/C_2 - 1] < 0$ for all λ on the contour Γ. Then by an obvious modification of Rouche's theorem,[10] it follows that $C(\lambda, \mu)$ and C_2 each have the same number of zeros within Γ. As C_2 has $k - 1$ zeros within Γ, this proves the lemma.

To show that $\text{Re}\,(C_1/C_2 - 1) < 0$ for all λ on Γ, let us consider separately the circular arc defined by (66b) and the radial lines defined by (66a) and (66c).

(*i*) On the circular arc (66b) straightforward manipulation gives

$$|C_1|^2 - \mu^{2+2/k}|C_2|^2 = -2\theta_2\mu^2(2 - \theta_1 - \theta_2)(1 - \cos k\theta) \leqq 0. \tag{69}$$

For $\mu < 1$, tberefore, $|C_1/C_2| < 1$, hence $\text{Re}\,(C_1/C_2 - 1) < 0$. If $\mu = 1$, this argument remains valid except at points where $\cos k\theta = 1$, for tben $|C_1/C_2| = 1$. However, if $\cos k\theta = 1$ and $\mu = 1$, we find that $C_1/C_2 - 1 = e^{j\theta} - 1$, whose real part <0 for $\pi/k \leqq \theta \leqq 2\pi - \pi/k$.

(*ii*) On the radial lines (66a) and (66c),

$$\text{Re}\left(\frac{C_1}{C_2} - 1\right) = \mu R\left(1 - \theta_2 - \frac{\mu\theta_1\theta_2}{R^k + \mu(1 - \theta_1)}\right)\cos\frac{\pi}{k} - 1, \tag{70}$$

which is obviously <0 for $R^k \leqq \mu$.

All the recursions of this paper except (63) correspond to the homogeneous form of (64), i.e., $\xi_i \equiv 0$. Solutions of all such recursions are of the form

$$\varphi_N = \sum_{i=0}^{k} \beta_i\lambda_i^N, \tag{71}$$

and therefore the asymptotic behavior is governed by λ_2, and λ_1 when it is equal to 1. (In the special case $\rho = 1$ and $\theta_1 = k\theta_2$, the dominant root is repeated and the usual modification must be made.) Dropping the subscript i from λ_i and β_i we give below an expression for the latter in terms of the initial conditions of the recursion, namely

$(\varphi_o, \cdots, \varphi_k)$:

$$\beta = a \cdot \left[\frac{-1}{\lambda^{k+1}} \cdot \frac{(1-\theta_1)}{(1-\theta_2)} \cdot \varphi_o + \left\{ 1 - \frac{1}{\lambda\rho(1-\theta_2)} \right\} \sum_{i=1}^{k-1} \frac{\varphi_i}{\lambda_i} + \frac{\varphi_k}{\lambda^k} \right], \tag{72a}$$

where

$$a = \rho(1-\theta_2)\lambda^{k+1}/[\lambda^k + k\rho^2(1-\theta_1-\theta_2) - (k+1)\rho(1-\theta_2)]. \tag{72b}$$

Thus, for example, the recursion for the probability of overflow [eq. (58)], with $\tau = 0$, in the canonical form (64) has the initial conditions $\varphi_0 = 1$, $\varphi_i = 1/[\rho(1-\theta_2)]^i$, $i = 1, 2, \cdots, k$. Also, in this case the dominant root of the characteristic polynonial λ_2 is the only root outside the unit circle in the complex plane. Therefore,

$$\frac{1}{G^{(N)}} \sim \beta\lambda_2^N, \tag{73}$$

where β is obtained from (72) for the appropriate values of $\varphi_o, \cdots, \varphi_k$. It can be easily shown that $\beta > 0$. In (73) (and similarly throughout this section) we use the notation $1/G^{(N)} \sim \beta\lambda_2^N$ to mean that $|1/G^{(N)} - \beta\lambda_2^N| < \epsilon^N$, for sufficiently large N, and $\epsilon < 1$.

In a manner similar to the derivation of (73) we can show that the probability of a transmission fault (Section 3.2) has the following asymptotic behavior

$$\frac{1}{T^{(N)}} \sim \alpha_1\lambda_2^N + \frac{\theta_1+\theta_2}{\theta_1 - k\theta_2} \quad \text{when } \theta_1 < k\theta_2 \tag{74a}$$

$$\sim \left[\frac{2\theta_2(N+1)}{2-\theta_1-\theta_2} + \frac{1-\theta_1-\theta_2}{1-\theta_1} \right] \frac{\theta_1+\theta_2}{\theta_1} \quad \text{when } \theta_1 = k\theta_2 \tag{74b}$$

$$\sim \frac{\theta_1+\theta_2}{\theta_1 - k\theta_2} \quad \text{when } \theta_1 > k\theta_2. \tag{74c}$$

In (74), α_1 is obtained from the generic formula (72a). We have shown that $\alpha_1 > 0$ and, of course, $1 < \lambda_2 < 1/1 - \theta_2$. Likewise, the mean first passage time (see Section 3.3) is, asymptotically,

$$F^{(N)} \sim \alpha_2\lambda_2^N - \frac{(\theta_1+\theta_2)N}{k\theta_2 - \theta_1}, \qquad \alpha_2 > 0, \qquad \text{when } \theta_1 < k\theta_2$$

$$\sim \frac{(\theta_1+\theta_2)N^2}{k(2-\theta_1-\theta_2)} + \alpha_3 N + \alpha_4, \qquad \text{when } \theta_1 = k\theta_2$$

$$\sim \frac{(\theta_1+\theta_2)N}{\theta_1 - k\theta_2} + \alpha_5, \qquad \text{when } \theta_1 > k\theta_2.$$

Finally,

$$H^{(N)} \sim \alpha_6 + \alpha_7 N, \qquad \alpha_7 > 0. \tag{76}$$

VI. COMPUTATIONS

We have written computer programs to recursively compute the quantities $T^{(N)}$, $F^{(N)}$, $G^{(N)}$, $H^{(N)}$ as functions of N for specified values of θ_1, θ_2, k, and ρ. Figures 3 through 6 are sample illustrations generated by these programs for $\theta_1 = 0.2$ and $\theta_2 = 0.1$. The asymptotic behavior

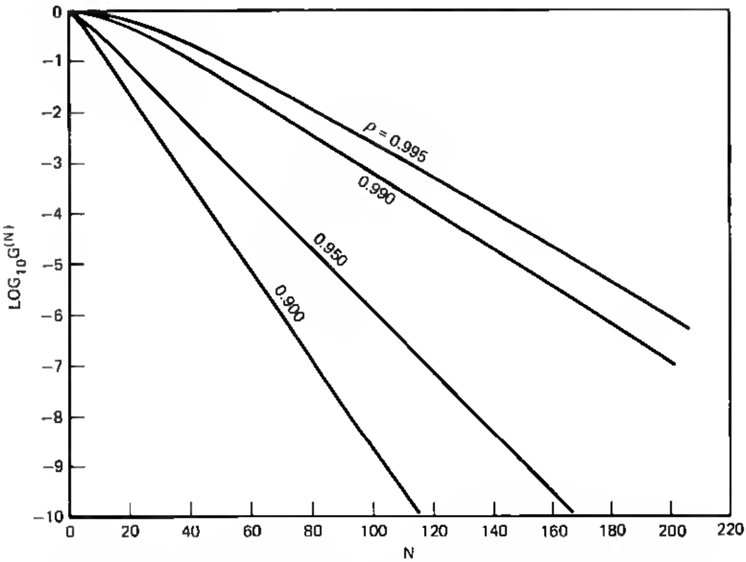

Fig. 3—Probability of overflow in a burst vs level ($\theta_1 = 0.2$, $\theta_2 = 0.1$, $k = 5$).

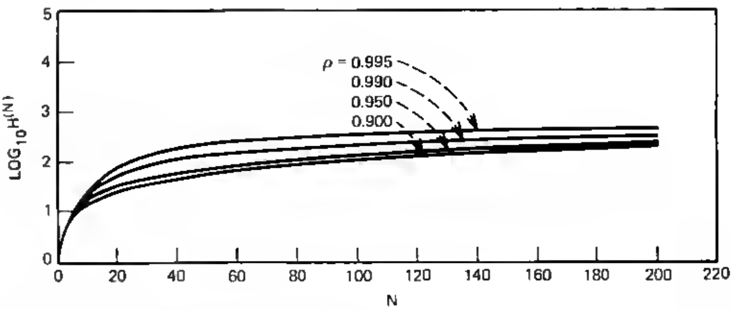

Fig. 4—Mean time for first passage conditional on overflow vs level ($\theta_1 = 0.2$, $\theta_2 = 0.1$, $k = 5$).

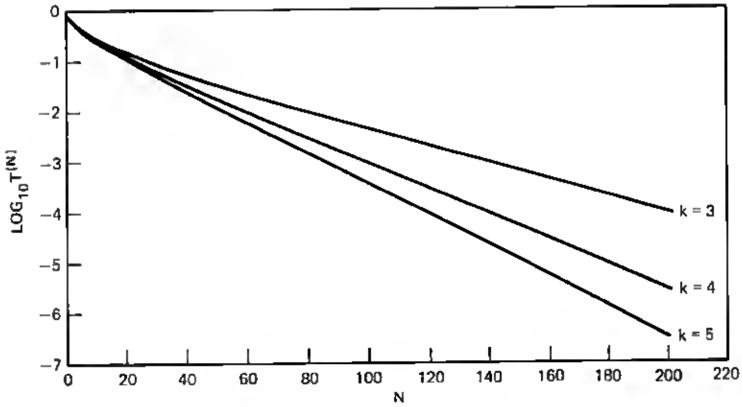

Fig. 5—Steady-state probability of transmission fault vs buffer size ($\theta_1 = 0.2$, $\theta_2 = 0.1$).

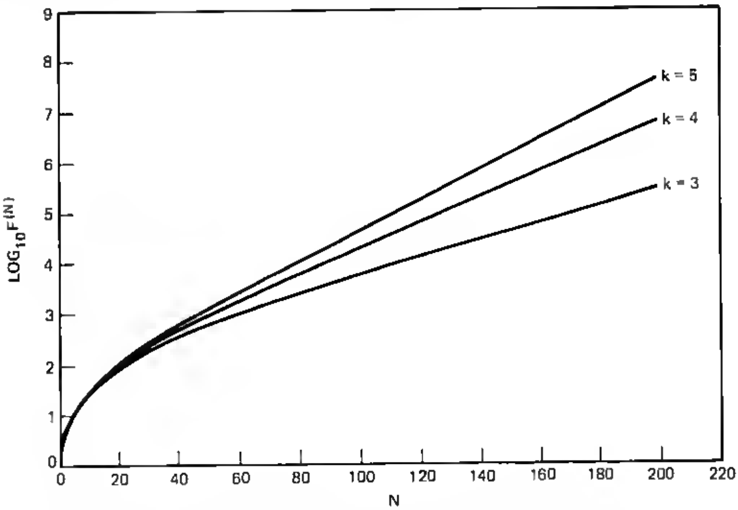

Fig. 6—Mean time for first passage in infinitely long bursts ($\theta_1 = 0.2$, $\theta_2 = 0.1$).

of the various quantities is seen to be in accord with that given by eqs. (73)–(76) of the previous section. The dependence on the parameters ρ and k also is intuitively reasonable.

APPENDIX

(a) We prove the assertion made in the text [immediately following eq. (37)] that the eigenvalues of the matrix (16) all lie strictly within

the unit circle. Let

$$\mathbf{M} \triangleq \begin{bmatrix} (1-\theta_1)\mathbf{B} - \lambda\mathbf{I} & \theta_2\mathbf{A} \\ \theta_1\mathbf{B} & (1-\theta_2)\mathbf{A} - \lambda\mathbf{I} \end{bmatrix}, \tag{77}$$

where $\mathbf{I}$ is the identity matrix of order $N + 1$. Then we must show that

$$\det \mathbf{M} \neq 0, \qquad \text{for } |\lambda| \geqq 1. \tag{78}$$

From the defining equations (14) and (18), we notice that the last column of A is identically zero. Thus

$$\det \mathbf{M} = -\lambda \det \mathbf{M}', \tag{79}$$

where $\mathbf{M}'$ is obtained from $\mathbf{M}$ by deleting its last row and column. Let m_{ij}, $i, j = 0, \cdots, 2N + 1$, denote the elements of $\mathbf{M}'$. Then a theorem of Hadamard[11] states that $\det \mathbf{M}' \neq 0$ provided $\mathbf{M}'$ is irreducible and

$$|m_{jj}| \geqq P_j = \sum_{i=0, i\neq j}^{2N+1} |m_{ij}|, \tag{80}$$

for all j, with strict inequality for at least one j. The irreducibility condition as stated in Ref. 11 is satisfied. To show (80) we note that

$$|m_{jj}| = |\lambda| \quad \text{and } P_j = 1, \quad \text{for } j = 1, \cdots, 2N + 1, \tag{81}$$

and

$$|m_{00}| = |1 - \theta_1 - \lambda|, \qquad P_0 = \theta_1. \tag{82}$$

Thus except at $\lambda = 1$, we find that (80) is true with strict inequality for $j = 0$. This proves the assertion (78) except for the point $\lambda = 1$. However, for $\lambda = 1$, $\det \mathbf{M} = \theta_1(1 - \theta_2)^N$ which, by assumption, $\neq 0$.

(b) Following eq. (52) we made the assertion that $\mathbf{I} - \rho\mathbf{C} - \rho^2\mathbf{D}$ is a nonsingular matrix for $\rho \leqq 1$. The proof is as follows. Let

$$\mathbf{M} \triangleq \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix}, \tag{83}$$

with $\mathbf{M}_{11} = (1 - \theta_1)\mathbf{B} - \lambda\mathbf{I}$, $\mathbf{M}_{21} = \theta_1\mathbf{B}$, etc. As $\mathbf{M}_{21}$ commutes with $\mathbf{M}_{11}$, an identity of Schur[12] states that

$$\det \mathbf{M} = \det [\mathbf{M}_{11}\mathbf{M}_{22} - \mathbf{M}_{21}\mathbf{M}_{12}]. \tag{84}$$

However, straightforward manipulation of the right side of (84) shows that

$$\det \mathbf{M} = \det (\lambda^2\mathbf{I} - \lambda\mathbf{C} - \mathbf{D}). \tag{85}$$

Then the assertion follows from (78).

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